

# CONCENTRATION OF MAPS AND GROUP ACTION

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**ABSTRACT.** In this paper, from the viewpoint of the concentration theory of maps, we study a compact group and a Lévy group action to a large class of metric spaces, such as  $\mathbb{R}$ -trees, doubling spaces, metric graphs, and Hadamard manifolds.

## 1. INTRODUCTION

Let a compact metric group  $G$  acts on a compact metric space  $X$ . In [16, Theorem 5.1], V. Milman considered a Hölder action (see Section 3.6.2 for the definition) and estimated the diameters of orbits from above by words of an isoperimetric property of the group  $G$  and a covering property of  $X$ . As he referred in the introduction, his idea came from the fixed point theory of a Lévy group action by M. Gromov and Milman in [7, Theorem 7.1] (see Section 4 for the definition of a Lévy group). In this paper, we consider general continuous actions of a compact metric group and a Lévy group to some concrete noncompact metric spaces, such as  $\mathbb{R}$ -trees, doubling spaces, metric graphs, and Hadamard manifolds.

Of isoperimetric inspiring, the Lévy-Milman concentration theory of maps played an important role in Milman's estimation (and also Gromov and Milman's theorem of a Lévy group action). Taking a point  $x \in X$ , he considered how concentrates the orbit map  $G \ni g \rightarrow gx \in X$  to a constant map. Recent developments of the concentration theory of maps by the author ([1], [2], [3]), by Gromov ([8], [10]), and by M. Ledoux and K. Oleszkiewicz ([12]) enable us to estimate how the orbit map concentrate to a constant map in the case where  $X$  is an  $\mathbb{R}$ -tree, a doubling space, a metric graph, and a Hadamard manifold. In stead of considering a Hölder action and a covering property, we provide an estimate of the diameters of orbits of a continuous action of a compact metric group to those metric spaces by words of the continuity of the action, an isoperimetric property of  $G$ , and a metric space property of  $X$ . Our results assert that we can measure how the action to those metric spaces is closed to the trivial action by the above words.

In the same point of view, we obtain two results of a Lévy group action to the above spaces. A Lévy group was first introduced and analyzed by Gromov and Milman in [7]. Gromov and Milman proved that every continuous action of a Lévy group to a

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compact metric space has a fixed point. They also pointed out that the unitary group  $U(\ell^2)$  of the separable Hilbert space  $\ell^2$  with the strong topology is a Lévy group. Many concrete examples of Lévy groups are known by the works of S. Glasner [6], H. Furstenberg and B. Weiss (unpublished), T. Giordano and V. Pestov [4], [5], and Pestov [20], [21]. For examples, groups of measurable maps from the standard Lebesgue measure space to compact groups, unitary groups of some von Neumann algebras, groups of measure and measure-class preserving automorphisms of the standard Lebesgue measure space, full groups of amenable equivalence relations, and the isometry groups of the universal Urysohn metric spaces are Lévy groups (see the recent monograph [18] for precise). One of our results states that there is no non-trivial uniformly continuous action of a Lévy group to the above spaces (Proposition 4.4). We also obtain a generalization of Gromov and Milman's fixed point theorem (Proposition 4.3). Both two results are obtained by making Gromov and Milman's argument precise.

The article is organized as follows. In Section 2, we recall basic facts about the concentration theory of maps and prepare for the Sections 3 and 4. In Section 3, we estimate the diameter of orbits of a compact group action to  $\mathbb{R}$ -trees, doubling spaces, metric graphs, and Hadamard manifolds. Section 4 is devoted to a Lévy group action to those spaces.

## 2. PRELIMINARIES

**2.1. Concentration function and observable diameter.** In this subsection, we recall some basic facts in the concentration theory of 1-Lipschitz maps. We recall relationships between an isoperimetric property of an mm-space (metric measure space) and the concentration theory of 1-Lipschitz functions. The concentration theory of 1-Lipschitz functions was introduced by Milman in his investigations of asymptotic geometric analysis ([13], [14], [15]). While the concentration theory of functions developed, the concentration theory of maps into general metric spaces was first studied by Gromov ([8], [9], [10]). He established the theory by introducing the observable diameter in [10]. We first recall its definition.

Let  $Y$  be a metric space and  $\nu$  a Borel measure on  $Y$  such that  $m := \nu(Y) < +\infty$ . We define for any  $\kappa > 0$

$$\text{diam}(\nu, m - \kappa) := \inf\{\text{diam } Y_0 \mid Y_0 \subseteq Y \text{ is a Borel subset such that } \nu(Y_0) \geq m - \kappa\}$$

and call it the *partial diameter* of  $\nu$ .

Let  $(X, d_X)$  be a complete separable metric space equipped with a finite Borel measure  $\mu_X$  on  $X$ . Henceforth, we call such a triple an *mm-space*.

**Definition 2.1** (Observable diameter). Let  $(X, d_X, \mu_X)$  be an mm-space with  $m_X := \mu_X(X)$  and  $Y$  a metric space. For any  $\kappa > 0$  we define the *observable diameter* of  $X$  by

$$\text{ObsDiam}_Y(X; -\kappa) := \sup\{\text{diam}(f_*(\mu_X), m_X - \kappa) \mid f : X \rightarrow Y \text{ is a 1-Lipschitz map}\},$$

where  $f_*(\mu_X)$  stands for the push-forward measure of  $\mu_X$  by  $f$ .

The idea of the observable diameter comes from the quantum and statistical mechanics, that is, we think of  $\mu_X$  as a state on a configuration space  $X$  and  $f$  is interpreted as an observable.

Given sequences  $\{X_n\}_{n=1}^\infty$  of mm-spaces and  $\{Y_n\}_{n=1}^\infty$  of metric spaces, observe that  $\lim_{n \rightarrow \infty} \text{ObsDiam}_{Y_n}(X_n; -\kappa) = 0$  for any  $\kappa > 0$  if and only if for any sequence  $\{f_n : X_n \rightarrow Y_n\}_{n=1}^\infty$  of 1-Lipschitz maps there exists a sequence  $\{m_{f_n}\}_{n=1}^\infty$  of points such that  $m_{f_n} \in Y_n$  and

$$\lim_{n \rightarrow \infty} \mu_{X_n}(\{x_n \in X_n \mid d_{Y_n}(f_n(x_n), m_{f_n}) \geq \varepsilon\}) = 0$$

for any  $\varepsilon > 0$ . A sequence  $\{X_n\}_{n=1}^\infty$  of mm-spaces is said to be a *Lévy family* if  $\lim_{n \rightarrow \infty} \text{ObsDiam}_{\mathbb{R}}(X_n; -\kappa) = 0$  for any  $\kappa > 0$ . The concept of Lévy families was first introduced in [7].

For an mm-space  $X$  with  $\mu_X(X) = 1$ , we define the *concentration function*  $\alpha_X : (0, +\infty) \rightarrow \mathbb{R}$  as the supremum of  $\mu_X(X \setminus A_{+r})$ , where  $A$  runs over all Borel subsets of  $X$  with  $\mu_X(A) \geq 1/2$  and  $A_{+r}$  is an open  $r$ -neighbourhood of  $A$ . This function describes an isoperimetric feature of the space  $X$ .

We shall consider each closed Riemannian manifold as an mm-space equipped with the volume measure normalized to have the total volume 1.

**Example 2.2.** Let  $M$  be a closed Riemannian manifold such that  $\text{Ric}_M \geq \tilde{\kappa}_1 > 0$ . By virtue of the Lévy-Gromov isoperimetric inequality, we obtain  $\alpha_M(r) \leq e^{-\tilde{\kappa}_1 r^2/2}$  (see [7, Section 1.2, Remark 2] or [11, Theorem 2.4]). Since  $\text{Ric}_{SO(n)} \geq (n-1)/4$ , we have  $\alpha_{SO(n)}(r) \leq e^{-(n-1)r^2/8}$  for example.

**Example 2.3.** Let  $M$  be a closed Riemannian manifold. We denote by  $\lambda_1(M)$  the non-zero first eigenvalue of the Laplacian on  $M$ . Then, for any  $r > 0$ , we have  $\alpha_M(r) \leq e^{-\sqrt{\lambda_1(M)}r/3}$  (see [7, Theorem 4.1] or [11, Theorem 3.1]). Since the  $n$ -dimensional torus  $\mathbb{T}^n := \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$  satisfies  $\lambda_1(\mathbb{T}^n) = \lambda_1(\mathbb{S}^1) = 1$ , we obtain  $\alpha_{\mathbb{T}^n}(r) \leq e^{-r/3}$  for example.

Let  $X$  be an mm-space and  $f : X \rightarrow \mathbb{R}$  a Borel measurable function. A number  $m_f \in \mathbb{R}$  is called a *median* of  $f$  if it satisfies that  $f_*(\mu_X)((-\infty, m_f]) \geq \mu_X/2$  and  $f_*(\mu_X)([m_f, +\infty)) \geq \mu_X/2$ . We remark that  $m_f$  does exist, but it is not unique in general.

Relationships between the concentration function and the observable diameter are the following:

**Lemma 2.4** (cf. [11, Section 1.3]). *Let  $X$  be an mm-space with  $\mu_X(X) = 1$ . Then, for any 1-Lipschitz function  $f : X \rightarrow \mathbb{R}$  and  $\varepsilon > 0$ , we have*

$$\mu_X(\{x \in X \mid |f(x) - m_f| \geq \varepsilon\}) \leq 2\alpha_X(\varepsilon).$$

**Lemma 2.5** (cf. [11, Section 1.3]). *Let  $X$  be an mm-space with  $\mu_X(X) = 1$ . Assume that a function  $\alpha : (0, +\infty) \rightarrow \mathbb{R}$  satisfies that*

$$\mu_X(\{x \in X \mid |f(x) - m_f| \geq \varepsilon\}) \leq \alpha(\varepsilon)$$

for any 1-Lipschitz function  $f : X \rightarrow \mathbb{R}$ . Then, we have  $\alpha_X(\varepsilon) \leq \alpha(\varepsilon)$ .

By Lemmas 2.4 and 2.5, we obtain the following corollary:

**Corollary 2.6** ([11, Section 1.3]). *A sequence  $\{X_n\}_{n=1}^\infty$  of mm-spaces is a Lévy family if and only if  $\lim_{n \rightarrow \infty} \alpha_{X_n}(r) = 0$  for any  $r > 0$ .*

Combining Lemma 2.4 with Examples 2.2 and 2.3, we obtain the following corollaries:

**Corollary 2.7.** *Let  $M$  be a closed Riemannian manifold such that  $\text{Ric}_M \geq \tilde{\kappa}_1 > 0$ . Then, for any  $\kappa > 0$ , we have*

$$\text{ObsDiam}_{\mathbb{R}}(M; -\kappa) \leq 2\sqrt{\frac{2 \log\left(\frac{2}{\kappa}\right)}{\tilde{\kappa}_1}}.$$

In particular, we have

$$\text{ObsDiam}_{\mathbb{R}}(SO(n); -\kappa) \leq 4\sqrt{\frac{2 \log\left(\frac{2}{\kappa}\right)}{n-1}}.$$

**Corollary 2.8.** *Let  $M$  be a closed Riemannian manifold. Then, for any  $\kappa > 0$ , we have*

$$\text{ObsDiam}_{\mathbb{R}}(M; -\kappa) \leq \frac{6 \log\left(\frac{2}{\kappa}\right)}{\sqrt{\lambda_1(M)}}.$$

In particular, we have

$$\text{ObsDiam}_{\mathbb{R}}(\mathbb{T}^n; -\kappa) \leq 6 \log\left(\frac{2}{\kappa}\right).$$

**2.2. Concentration and separation.** In this section, we recall the notion of the separation distance for an mm-space which was introduced in [10]. We review relationships between the observable diameter and the separation distance. The separation distance plays an important role throughout this paper.

Let  $X$  be an mm-space. For  $\kappa_1, \kappa_2 \geq 0$ , we define the *separation distance*  $\text{Sep}(X; \kappa_1, \kappa_2) = \text{Sep}(\mu_X; \kappa_1, \kappa_2)$  of  $X$  as the supremum of the distance  $d_X(A, B)$ , where  $A$  and  $B$  are Borel subsets of  $X$  satisfying that  $\mu_X(A) \geq \kappa_1$  and  $\mu_X(B) \geq \kappa_2$ .

Relationships between the observable diameter and the separation distance are followings. We refer to [2, Subsection 2.2] for precise proofs.

**Lemma 2.9** (cf. [10, Section 3 $\frac{1}{2}$ .33]). *Let  $X$  be an mm-space and  $\kappa, \kappa' > 0$  with  $\kappa > \kappa'$ . Then we have*

$$\text{ObsDiam}_{\mathbb{R}}(X; -\kappa') \geq \text{Sep}(X; \kappa, \kappa).$$

**Remark 2.10.** In [10, Section 3 $\frac{1}{2}$ .33], Lemma 2.9 is stated as  $\kappa = \kappa'$ , but that is not true in general. For example, let  $X := \{x_1, x_2\}$ ,  $d_X(x_1, x_2) := 1$ , and  $\mu_X(\{x_1\}) = \mu_X(\{x_2\}) := 1/2$ . Putting  $\kappa = \kappa' = 1/2$ , we have  $\text{ObsDiam}_{\mathbb{R}}(X; -1/2) = 0$  and  $\text{Sep}(X; 1/2, 1/2) = 1$ .

**Lemma 2.11** (cf. [10, Section 3 $\frac{1}{2}$ .33]). *Let  $\nu$  be a Borel measure on  $\mathbb{R}$  with  $m := \nu(\mathbb{R}) < +\infty$ . Then, for any  $\kappa > 0$  we have*

$$\text{diam}(\nu, m - 2\kappa) \leq \text{Sep}(\nu; \kappa, \kappa).$$

*In particular, for any  $\kappa > 0$  we have*

$$\text{ObsDiam}_{\mathbb{R}}(X; -2\kappa) \leq \text{Sep}(X; \kappa, \kappa).$$

**Corollary 2.12** (cf. [10, Section 3 $\frac{1}{2}$ .33]). *A sequence  $\{X_n\}_{n=1}^{\infty}$  of mm-spaces is a Lévy family if and only if  $\lim_{n \rightarrow \infty} \text{Sep}(X_n; \kappa, \kappa) = 0$  for any  $\kappa > 0$ .*

**2.3. Compact metric group action and diameter of a measure.** Let a compact metric group  $G$  continuously acts on a metric space  $X$ . For each  $\eta > 0$ , we define a (possibly infinite) number  $\rho(\eta) = \rho^{(G, X)}(\eta)$  as the supremum of  $d_X(gx, gy)$  for all  $g \in G$  and  $x, y \in X$  with  $d_X(x, y) \leq \eta$ . Given a point  $x \in X$ , we indicate by  $f_x : G \rightarrow X$  the orbit map of  $x$ , that is,  $f_x(g) := gx$  for any  $g \in G$ . For the Haar measure  $\mu_G$  on  $G$  normalized as  $\mu_G(G) = 1$ , we put  $\nu_{G, x} := (f_x)_*(\mu_G)$ .

**Proposition 2.13.** *Assume that  $\nu_{G, x}(B_X(y, \delta)) > 1/2$  for some  $y \in X$  and  $\delta > 0$ . Then, we have*

$$(2.1) \quad d_X(y, gy) \leq \delta + \rho(\delta)$$

*for any  $g \in G$ . Moreover, there exists a point  $x_0 \in Gx$  such that*

$$(2.2) \quad d_X(x_0, gx_0) \leq \min\{2\delta + \rho(2\delta), 2\delta + 2\rho(\delta)\}$$

*for any  $g \in G$ .*

*Proof.* Taking any  $g \in G$ , we first prove (2.1). Since  $gB_X(y, \delta) \subseteq B_X(gy, \rho(\delta))$  and the measure  $\nu_{G, x}$  is  $G$ -invariant, from the assumption, we have

$$\nu_{G, x}(B_X(gy, \rho(\delta))) \geq \nu_{G, x}(gB_X(y, \delta)) = \nu_{G, x}(B_X(y, \delta)) > 1/2.$$

Combining this with  $\nu_{G, x}(B_X(y, \eta)) > 1/2$ , we get  $\nu_{G, x}(B_X(y, \delta) \cap B_X(gy, \rho(\delta))) > 0$ , which implies (2.1).

We next prove (2.2). Since the orbit  $Gx$  is compact, the support of the measure  $\nu_{G, x}$  is included in  $Gx$ . Hence, there exists a point  $x_0 \in B_X(y, \delta) \cap Gx$ . Let  $g \in G$ . Since  $\nu_{G, x}(B_X(x_0, 2\delta)) \geq \nu_{G, x}(B_X(x_0, 2\delta)) > 0$ , by using (2.1), we obtain  $d_X(x_0, gx_0) \leq 2\delta + \rho(2\delta)$ . We also have

$$\begin{aligned} d_X(x_0, gx_0) &\leq d_X(x_0, y) + d_X(y, gy) + d_X(gy, gx_0) \\ &\leq \delta + (\delta + \rho(\delta)) + \rho(\delta) \\ &= 2\delta + 2\rho(\delta), \end{aligned}$$

which implies (2.2). This completes the proof.  $\square$

**Proposition 2.14.** *Assume that  $\nu_{G,x}(A) > 1/2$  for some Borel subset  $A \subseteq X$ . Then, there exists a point  $x_0 \in Gx$  such that*

$$d_X(x_0, gx_0) \leq \text{diam } A + \rho(\text{diam } A)$$

for any  $g \in G$ .

*Proof.* Since  $A \cap Gx \neq \emptyset$ , the claim follows from the same argument in the proof of Proposition 2.13.  $\square$

For any  $\eta > 0$ , we put  $\rho(+\eta) := \lim_{\eta' \downarrow \eta} \omega_x(\eta')$ .

**Corollary 2.15.** *There exists a point  $z_x \in Gx$  such that*

$$d_X(z_x, gz_x) \leq \lim_{\kappa \uparrow 1/2} \text{diam}(\nu_{G,x}, 1 - \kappa) + \rho\left(+ \lim_{\kappa \uparrow 1/2} \text{diam}(\nu_{G,x}, 1 - \kappa)\right)$$

for any  $g \in G$ .

For any  $\eta > 0$ , we define a (possibly infinite) number  $\omega_x(\eta) = \omega_x^{(G,X)}(\eta)$  as the supremum of  $d_X(gx, g'x)$  for all  $g, g' \in G$  with  $d_G(g, g') \leq \eta$ .

**Lemma 2.16.** *For any  $\kappa_1, \kappa_2 > 0$ , we have*

$$\text{Sep}(\nu_{G,x}; \kappa_1, \kappa_2) \leq \omega_x(+ \text{Sep}(G; \kappa_1, \kappa_2)).$$

*Proof.* Let  $A$  and  $B$  be two Borel subsets such that  $\nu_{G,x}(A) \geq \kappa_1$  and  $\nu_{G,x}(B) \geq \kappa_2$ . Since  $\mu_G((f_x)^{-1}(A)) \geq \kappa_1$  and  $\mu_G((f_x)^{-1}(B)) \geq \kappa_2$ , we have  $d_G((f_x)^{-1}(A), (f_x)^{-1}(B)) \leq \text{Sep}(G; \kappa_1, \kappa_2)$ . Thus, from the definition of  $\omega_x$ , we obtain  $d_X(A, B) \leq \omega_x(+ \text{Sep}(G; \kappa_1, \kappa_2))$ . This completes the proof.  $\square$

**Corollary 2.17** (cf. [7, Section 5.2]). *Assume that a sequence  $\{G_n\}_{n=1}^\infty$  of compact metric groups is a Lévy family and each  $G_n$  acts on a metric space  $X$ . Assume also that there exist a sequence  $\{x_n\}_{n=1}^\infty$  of points in  $X$  and a function  $\omega : (0, +\infty) \rightarrow [0, +\infty]$  such that  $\lim_{\eta \rightarrow 0} \omega(\eta) = 0$  and  $\omega_{x_n}^{(G_n, X)}(\eta) \leq \omega(\eta)$  for any  $n \in \mathbb{N}$  and  $\eta > 0$ . Then, the sequence  $\{(X, d_X, \nu_{G_n, x_n})\}_{n=1}^\infty$  of mm-spaces is a Lévy family.*

### 3. ESTIMATES OF THE DIAMETERS OF ORBITS

Throughout this section, we always assume that a compact metric group  $G$  continuously acts on a metric space  $X$ . We shall consider the group  $G$  as an mm-space  $(G, d_G, \mu_G)$ , where  $\mu_G$  is the Haar measure on  $G$  normalized as  $\mu_G(G) = 1$ . In this section, motivated by the work of Milman [16], we shall estimate the diameters of orbits  $Gx$  from above for concrete metric spaces  $X$  by words of the continuity of the action, an isoperimetric property of  $G$ , and a metric space property of  $X$ . For this purpose, we use the notation  $\rho = \rho^{(G,X)}$  and  $\omega_x = \omega_x^{(G,X)}$  defined in Subsection 2.3. We first consider the case where the orbit map  $f_x : G \ni g \mapsto gx \in X$  for some  $x \in X$  is a 1-Lipschitz map. In this case, applying Corollary 2.15, we obtain the following:

**Proposition 3.1.** *For any  $\kappa \in (0, 1/2)$ , there exists a point  $z_\kappa \in X$  such that*

$$d_X(z_\kappa, gz_\kappa) \leq \text{ObsDiam}_X(G; -\kappa) + \rho(\text{ObsDiam}_X(G; -\kappa))$$

*for any  $g \in G$ .*

**3.1. Case of Euclidean spaces.** In this subsection, we consider the case where the metric space  $X$  is the Euclidean space  $\mathbb{R}^k$ . Let  $\text{pr}_i : \mathbb{R}^k \ni x = (x_i)_{i=1}^k \mapsto x_i \in \mathbb{R}$  be the projection.

**Proposition 3.2** (cf. [10, Section 3 $\frac{1}{2}$ .32]). *For any finite Borel measure  $\nu$  on  $\mathbb{R}^k$  with  $m := \nu(\mathbb{R}^k)$ , we have*

$$\text{diam}(\nu, m - \kappa) \leq \sqrt{k} \max_{1 \leq i \leq k} \text{diam}\left((\text{pr}_i)_*(\nu), m - \frac{\kappa}{k}\right).$$

Applying Corollary 2.12 to Proposition 3.2, we obtain the following corollary:

**Corollary 3.3** (cf. [10, Section 3 $\frac{1}{2}$ .32]). *For any Lévy family  $\{X_n\}_{n=1}^\infty$  and any  $\kappa > 0$ , we have*

$$\lim_{n \rightarrow \infty} \text{ObsDiam}_{\mathbb{R}^k}(X_n; -\kappa) = 0.$$

**Proposition 3.4.** *Assume that a compact metric group  $G$  continuously acts on the Euclidean space  $\mathbb{R}^k$  and put  $r := \lim_{\kappa \uparrow 1/(4k)} \text{Sep}(G; \kappa, \kappa)$ . Then, for any  $x \in \mathbb{R}^k$ , there exists a point  $z_x \in Gx$  such that*

$$(3.1) \quad d_{\mathbb{R}^k}(z_x, gz_x) \leq \sqrt{k}\omega_x(+r) + \rho(+\sqrt{k}\omega_x(+r))$$

*for any  $g \in G$ .*

*Proof.* Combining Lemma 2.16 with Proposition 3.2, we get

$$\begin{aligned} \text{diam}(\nu_{G,x}, 1 - \kappa) &\leq \sqrt{k} \max_{1 \leq i \leq k} \text{diam}\left((\text{pr}_i)_*(\nu_{G,x}), 1 - \frac{\kappa}{k}\right) \\ &\leq \sqrt{k} \max_{1 \leq i \leq k} \text{Sep}\left((\text{pr}_i)_*(\nu_{G,x}); \frac{\kappa}{2k}, \frac{\kappa}{2k}\right) \\ &\leq \sqrt{k} \text{Sep}\left(\nu_{G,x}; \frac{\kappa}{2k}, \frac{\kappa}{2k}\right) \\ &\leq \sqrt{k}\omega_x\left(+\text{Sep}\left(G; \frac{\kappa}{2k}, \frac{\kappa}{2k}\right)\right). \end{aligned}$$

Applying this to Corollary 2.15, we obtain (3.1). This completes the proof.  $\square$

**3.2. Case of compact metric spaces.** In this subsection, we treat the case where the metric space  $X$  is a compact metric space  $K$ . For any  $\delta > 0$ , we denote by  $N_K(\delta)$  the minimum number of Borel subsets of diameter at most  $\delta$  which cover  $K$ .

**Proposition 3.5** (cf. [10, Section 3 $\frac{1}{2}$ .34]). *For any  $\delta, \kappa > 0$  and any finite Borel measure  $\nu$  on  $K$  with  $m := \nu(K)$ , we have*

$$\text{diam}(\nu, m - \kappa) \leq \text{Sep}\left(\nu; \frac{\kappa}{N_K(\delta)}, \frac{\kappa}{N_K(\delta)}\right) + 2\delta.$$

**Corollary 3.6** (cf. [10, Section 3 $\frac{1}{2}$ .34]). *Let  $\{X_n\}_{n=1}^\infty$  be a Lévy family and  $K$  a compact metric space. Then, for any  $\kappa > 0$ , we have*

$$\lim_{n \rightarrow \infty} \text{ObsDiam}_K(X_n; -\kappa) = 0.$$

By virtue of Proposition 3.5, the same proof of Proposition 3.4 yields the following proposition:

**Proposition 3.7.** *Assume that a compact metric group  $G$  continuously acts on a compact metric space  $K$  and put  $r_{x,\delta} := \omega_x(+\lim_{\kappa \uparrow 1/(2N_K(\delta))} \text{Sep}(G; \kappa, \kappa)) + 2\delta$  for  $x \in K$  and  $\delta > 0$ . Then, there exists a point  $z_{x,\delta} \in Gx$  such that*

$$d_K(z_{x,\delta}, gz_{x,\delta}) \leq r_{x,\delta} + \rho(+r_{x,\delta})$$

for any  $g \in G$ .

Proposition 3.7 generalizes Milman's result [16, Theorem 5.1].

**3.3. Case of  $\mathbb{R}$ -trees.** In this subsection, we consider the case where the metric space  $X$  is an  $\mathbb{R}$ -tree  $T$ . For this purpose, we first recall some standard terminologies in metric geometry. Let  $(X, d_X)$  be a metric space. A rectifiable curve  $\gamma : [0, 1] \rightarrow X$  is called a *geodesic* if its arclength coincides with the distance  $d_X(\gamma(0), \gamma(1))$  and it has a constant speed, i.e., parameterized proportionally to the arc length. We say that  $(X, d_X)$  is a *geodesic space* if any two points in  $X$  are joined by a geodesic between them.

A complete metric space  $T$  is called an  *$\mathbb{R}$ -tree* if it has the following properties:

- (1) Any two points in  $T$  are connected by a unique unit speed geodesic.
- (2) The image of every simple path in  $T$  is the image of a geodesic.

To answer Gromov's exercise in [10, Section 3 $\frac{1}{2}$ .32], the author proved the following theorem:

**Theorem 3.8** (cf. [1, Proposition 5.1]). *For any  $\kappa > 0$  and finite Borel measure  $\nu$  on  $T$  with  $m := \nu(T)$ , we have*

$$\nu\left(B_T\left(x_\nu, \text{Sep}\left(\nu; \frac{\kappa}{2}, \frac{m}{3}\right)\right)\right) \geq 1 - \kappa.$$

**Corollary 3.9** (cf. [1, Theorem 1.1]). *Let  $\{X_n\}_{n=1}^\infty$  be a Lévy family and  $T$  an  $\mathbb{R}$ -tree. Then, for any  $\kappa > 0$ , we have*

$$\lim_{n \rightarrow \infty} \text{ObsDiam}_T(X_n; -\kappa) = 0.$$

By Proposition 2.13 and Theorem 3.8, the following proposition follows from the same proof of Proposition 3.4.

**Proposition 3.10.** *Assume that a compact metric group  $G$  continuously acts on an  $\mathbb{R}$ -tree  $T$ . Then, for any  $x \in T$  and  $\kappa \in (0, 1/4)$ , there exists a point  $z_{x,\kappa} \in T$  such that*

$$d_T(z_{x,\kappa}, gz_{x,\kappa}) \leq \omega_x\left(+\text{Sep}\left(G; \kappa, \frac{1}{3}\right)\right) + \rho\left(\omega_x\left(+\text{Sep}\left(G; \kappa, \frac{1}{3}\right)\right)\right)$$



for any  $g \in G$ . Put  $r := \lim_{\kappa \uparrow 1/4} \text{Sep}(G; \kappa, \kappa)$ . Then, there also exists a point  $z_x \in Gx$  such that

$$d_T(z_x, gz_x) \leq \min\{2\omega_x(+r) + \rho(+2\omega_x(+r)), 2\omega_x(+r) + 2\rho(\omega_x(+r))\}$$

for any  $g \in G$ .

**3.4. Case of doubling spaces.** Throughout this subsection, we consider the case where the metric space  $X$  is a doubling space. A complete metric space  $X$  is called a *doubling space* if there exist  $R_1 > 0$  and a function  $D = D_X : (0, R_1] \rightarrow (0, +\infty)$  satisfying the following condition: Every closed ball with radius  $2r_1 \leq 2R_1$  is covered by at most  $D(r_1)$  closed balls with radius  $r_1$ . This condition is equivalent to the following condition: There exists a function  $C = C_X = C(r_1, r_2) : (0, 2R_1] \times (0, 2R_1] \rightarrow (0, +\infty)$  such that for every  $(r_1, r_2) \in (0, 2R_1] \times (0, 2R_1]$ , every  $r_1$ -separated subset in any closed ball in  $X$  with radius  $r_2$  contains at most  $C(r_1, r_2)$  elements. For example, a complete Riemannian manifold with Ricci curvature bounded from below is a doubling space (see the proof of Corollary 3.20).

Although the proof of the following theorem is the same analogue to [2, Theorem 1.3], we give it for completeness.

**Theorem 3.11.** *Let  $X$  be a doubling space and  $\nu$  a finite Borel measure on  $X$  with  $m := \nu(X)$ . Assume that a positive number  $r_0$  satisfies*

$$r_0 > \max \left\{ \text{Sep} \left( \nu; \kappa, \frac{m}{C(r_0, 5r_0)} \right), \text{Sep} \left( \nu; \frac{m - \kappa}{3}, \frac{m - \kappa}{3} \right), \text{Sep} \left( \nu; \frac{m - \kappa}{3}, \kappa \right) \right\}$$

for some  $\kappa > 0$ . Then there exists a point  $x_0 \in X$  such that  $\nu(B_X(x_0, 3r_0)) \geq m - \kappa$ .

*Proof.* Take a maximal  $r_0$ -separated set  $\{\xi_\alpha\}_{\alpha \in \mathcal{A}}$  of  $X$ . From the doubling property of  $X$ , there exists  $\alpha_0 \in \mathcal{A}$  such that

$$k := \#\{\beta \in \mathcal{A} \mid \xi_\beta \in B_X(\xi_{\alpha_0}, 5r_0)\} = \max_{\alpha \in \mathcal{A}} \#\{\beta \in \mathcal{A} \mid \xi_\beta \in B_X(\xi_\alpha, 5r_0)\} \leq C(r_0, 5r_0).$$

Putting  $\{\beta_1, \beta_2, \dots, \beta_k\} := \{\beta \in \mathcal{A} \mid \xi_\beta \in B_X(\xi_{\alpha_0}, 5r_0)\}$ , we take a subset  $J_1 \subseteq \{\xi_\alpha\}_{\alpha \in \mathcal{A}}$  which is maximal with respect to the properties that  $J_1$  is  $5r_0$ -separated and  $\xi_{\beta_1} \in J_1$ ,  $\xi_{\beta_2} \notin J_1, \dots, \xi_{\beta_k} \notin J_1$ . We then take  $J_2 \subseteq \{\xi_\alpha\}_{\alpha \in \mathcal{A}} \setminus J_1$  which is maximal with respect to the properties that  $J_2$  is  $5r_0$ -separated and  $\xi_{\beta_2} \in J_2$ ,  $\xi_{\beta_3} \notin J_2, \dots, \xi_{\beta_k} \notin J_2$ . In the same way, we pick  $J_3 \subseteq \{\xi_\alpha\}_{\alpha \in \mathcal{A}} \setminus (J_1 \cup J_2), \dots, J_k \subseteq \{\xi_\alpha\}_{\alpha \in \mathcal{A}} \setminus (J_1 \cup J_2 \cup \dots \cup J_{k-1})$ . We then have

**Claim 3.12.**  $\{\xi_\alpha\}_{\alpha \in \mathcal{A}} = J_1 \cup J_2 \cup \dots \cup J_k$ .

*Proof.* Suppose that  $\xi_\alpha \notin J_1 \cup J_2 \cup \dots \cup J_k$  for some  $\alpha \in \mathcal{A}$ . Since each  $J_i$  is maximal, there exists  $\xi_{\alpha_i} \in J_i$  such that  $d_X(\xi_\alpha, \xi_{\alpha_i}) < 5r_0$  and  $\xi_\alpha \neq \xi_{\alpha_i}$ . We therefore obtain

$$k + 1 \leq \#\{\xi_\alpha, \xi_{\alpha_1}, \xi_{\alpha_2}, \dots, \xi_{\alpha_k}\} \leq \#\{\beta \in \mathcal{A} \mid \xi_\beta \in B_X(\xi_\alpha, 5r_0)\} \leq k,$$

which is a contradiction. This completes the proof of the claim.  $\square$

By Claim 3.12, we have  $X = \bigcup_{i=1}^k \bigcup_{\xi_\alpha \in J_i} B_X(\xi_\alpha, r_0)$ . Hence there exists  $i$ ,  $1 \leq i \leq k$  such that

$$\nu\left(\bigcup_{\xi_\alpha \in J_i} B_X(\xi_\alpha, r_0)\right) \geq \frac{m}{k} \geq \frac{m}{C(r_0, 5r_0)}.$$

We then have

**Claim 3.13.**

$$\nu\left(\bigcup_{\xi_\alpha \in J_i} B_X(\xi_\alpha, 2r_0)\right) \geq m - \kappa.$$

*Proof.* Supposing that  $\nu(\bigcup_{\xi_\alpha \in J_i} B_X(\xi_\alpha, 2r_0)) < m - \kappa$ , from the assumption of  $r_0$ , we have

$$r_0 \leq d_X\left(X \setminus \bigcup_{\xi_\alpha \in J_i} B_X(\xi_\alpha, 2r_0), \bigcup_{\xi_\alpha \in J_i} B_X(\xi_\alpha, r_0)\right) \leq \text{Sep}\left(\nu; \kappa, \frac{m}{C(r_0, 5r_0)}\right) < r_0.$$

This is a contradiction. This completes the proof of the claim.  $\square$

**Claim 3.14.** *There exists  $\xi_\gamma \in J_i$  such that  $\nu(B_X(\xi_\gamma, 2r_0)) \geq (m - \kappa)/3$ .*

*Proof.* Suppose that  $\nu(B_X(\xi_\alpha, 2r_0)) < (m - \kappa)/3$  for any  $\xi_\alpha \in J_i$ . Then, by Claim 3.13, there exists  $J'_i \subseteq J_i$  such that

$$\frac{m - \kappa}{3} \leq \nu\left(\bigcup_{\xi_\alpha \in J'_i} B_X(\xi_\alpha, 2r_0)\right) < \frac{2(m - \kappa)}{3}.$$

Thus, putting  $J''_i := J_i \setminus J'_i$ , from the assumption of  $r_0$ , we get

$$r_0 \leq d_X\left(\bigcup_{\xi_\alpha \in J'_i} B_X(\xi_\alpha, 2r_0), \bigcup_{\xi_\alpha \in J''_i} B_X(\xi_\alpha, 2r_0)\right) \leq \text{Sep}\left(\nu; \frac{m - \kappa}{3}, \frac{m - \kappa}{3}\right) < r_0.$$

This is a contradiction. This completes the proof of the claim.  $\square$

Combining Claim 3.14 with the same method of the proof of Claim 3.13, we finally obtain  $\nu(B_X(\xi_\gamma, 3r_0)) \geq 1 - \kappa$ . This completes the proof of the theorem.  $\square$

By Corollary 2.12 and Theorem 3.11, we get the following corollary:

**Corollary 3.15** (cf. [2, Theorem 1.3]). *Let  $\{X_n\}_{n=1}^\infty$  be a Lévy family and  $X$  a doubling space. Then, for any  $\kappa > 0$ , we have*

$$\lim_{n \rightarrow \infty} \text{ObsDiam}_X(X_n; -\kappa) = 0.$$

Applying Theorem 3.11 to Proposition 2.13, we obtain the following proposition:

**Proposition 3.16.** *Let a compact metric group  $G$  continuously acts on a doubling space  $X$ . Assume that a positive number  $r_0$  satisfies*

$$r_0 > \max\left\{\omega_x\left(+\text{Sep}\left(\nu; \kappa, \frac{1}{C(r_0, 5r_0)}\right)\right), \omega_x\left(+\text{Sep}\left(\nu; \frac{1 - \kappa}{3}, \frac{1 - \kappa}{3}\right)\right), \omega_x\left(+\text{Sep}\left(\nu; \frac{1 - \kappa}{3}, \kappa\right)\right)\right\}$$

for some  $\kappa \in (0, 1/2)$ . Then there exists a point  $z_{x,\kappa} \in X$  such that

$$d_X(z_{x,\kappa}, gz_{x,\kappa}) \leq 3r_0 + \rho(3r_0)$$

for any  $g \in G$ . Moreover, there exists a point  $z'_{x,\kappa} \in Gx$  such that

$$d_X(z'_{x,\kappa}, gz'_{x,\kappa}) \leq \min\{6r_0 + \rho(6r_0), 6r_0 + 2\rho(3r_0)\}$$

for any  $g \in G$ .

We next consider the case where the function  $D = D_X : (0, +\infty) \rightarrow (0, +\infty)$  is a constant function. This is equivalent to the following condition: The function  $C = C_X : (0, +\infty) \times (0, +\infty) \rightarrow (0, +\infty)$  satisfies that  $C(\alpha r, \alpha s) = C(r, s)$  for any  $r, s, \alpha > 0$ . We call such a metric space a *large scale doubling space*.

By Theorem 3.11, we obtain the following corollary:

**Corollary 3.17.** *Let  $X$  be a large scale doubling space and  $\nu$  be a finite Borel measure on  $X$  with  $m := \nu(X)$  and put*

$$r_\kappa := \max \left\{ \text{Sep} \left( \nu; \kappa, \frac{m}{C(1, 5)} \right), \text{Sep} \left( \nu; \frac{m - \kappa}{3}, \frac{m - \kappa}{3} \right), \text{Sep} \left( \nu; \frac{m - \kappa}{3}, \kappa \right) \right\}$$

for  $\kappa > 0$ . Then, there exists a point  $x_\kappa \in X$  such that  $\nu(B_X(x_\kappa, 3r_\kappa)) \geq m - \kappa$ .

Applying Corollary 3.17 to Proposition 2.13, we obtain the following proposition:

**Proposition 3.18.** *Assume that a compact metric group  $G$  continuously acts on a large scale doubling space  $X$ . Put*

$$r_{x,\kappa} := \max \left\{ \omega_x \left( + \text{Sep} \left( G; \kappa, \frac{1}{C(1, 5)} \right) \right), \omega_x \left( + \text{Sep} \left( G; \frac{1 - \kappa}{3}, \frac{1 - \kappa}{3} \right) \right), \right. \\ \left. \omega_x \left( + \text{Sep} \left( G; \frac{1 - \kappa}{3}, \kappa \right) \right) \right\}$$

for  $x \in X$  and  $\kappa > 0$ . Then, for any  $\kappa \in (0, 1/2)$ , there exists a point  $z_{x,\kappa} \in X$  such that

$$d_X(z_{x,\kappa}, gz_{x,\kappa}) \leq 3r_{x,\kappa} + \rho(3r_{x,\kappa})$$

for any  $g \in G$ . There also exists a point  $z'_{x,\kappa} \in Gx$  such that

$$d_X(z'_{x,\kappa}, gz'_{x,\kappa}) \leq \min\{6r_{x,\kappa} + \rho(6r_{x,\kappa}), 6r_{x,\kappa} + 2\rho(3r_{x,\kappa})\}$$

for any  $g \in G$ .

Assume that a complete metric space  $X$  has a doubling measure  $\nu_X$ , that is,  $\nu_X$  is a (not only finite) Borel measure on  $X$  having the following properties:  $X = \text{Supp } \nu_X$  and there exists a constant  $C = C(X) > 0$  such that

$$(3.2) \quad \nu_X(B_X(x, 2r)) \leq C\nu_X(B_X(x, r))$$

for any  $x \in X$  and  $r > 0$ . For example, by virtue of the Bishop-Gromov volume comparison theorem, the volume measure of an  $n$ -dimensional complete Riemannian manifold  $M$  with nonnegative Ricci curvature is a doubling measure with  $C(M) = 2^n$ .

**Lemma 3.19** (cf. [2, Lemma 2.1]). *Let  $(X, \nu_X)$  be a complete metric space with a doubling measure  $\nu_X$ . Then, for any  $0 < r_1 \leq r_2$  and  $x, y \in X$  with  $x \in B_X(y, r_2)$ , we have*

$$\frac{\nu_X(B_X(x, r_1))}{\nu_X(B_X(y, r_2))} \geq \frac{1}{C^2} \left( \frac{r_1}{r_2} \right)^{\log_2 C} = C^{\log_2 \frac{r_1}{r_2} - 2}.$$

**Corollary 3.20.** *The space  $(X, \nu_X)$  is a large scale doubling space with  $C_X(r_1, r_2) \leq C^{2+\log_2\{(r_1+2r_2)/r_1\}}$ . In particular, we have  $C_X(1, 5) \leq C^{2+\log_2 11}$ .*

*Proof.* Given any  $x \in X$  and  $r_1, r_2 > 0$  with  $r_2 \geq r_1$ , we let  $\{\xi_\alpha\}_{\alpha \in \mathcal{A}} \subseteq B_X(x, r_2)$  be an arbitrary  $r_1$ -separated set. Note that closed balls  $B_X(\xi_\alpha, 2^{-1}r_1 - \varepsilon)$  are mutually disjoint for any  $\varepsilon > 0$ . We hence have

$$\begin{aligned} \nu_X(B_X(x, 2^{-1}r_1 + r_2)) &\geq \nu_X\left(\bigcup_{\alpha \in \mathcal{A}} B_X(\xi_\alpha, 2^{-1}r_1 - \varepsilon)\right) \\ &= \sum_{\alpha \in \mathcal{A}} \nu_X(B_X(\xi_\alpha, 2^{-1}r_1 - \varepsilon)) \\ &\geq \nu_X(B_X(\xi_{\alpha_0}, 2^{-1}r_1 - \varepsilon)) \# \mathcal{A}, \end{aligned}$$

where  $\nu_X(B_X(\xi_{\alpha_0}, 2^{-1}r_1 - \varepsilon)) = \min_{\alpha \in \mathcal{A}} \nu_X(B_X(\xi_\alpha, 2^{-1}r_1 - \varepsilon))$ . Applying this to Lemma 3.19, we obtain

$$\# \mathcal{A} \leq \frac{\nu_X(B_X(x, 2^{-1}r_1 + r_2))}{\nu_X(B_X(\xi_{\alpha_0}, 2^{-1}r_1 - \varepsilon))} \leq C^{2+\log_2\{(r_1+2r_2)/(r_1-2\varepsilon)\}}.$$

This completes the proof.  $\square$

Combining Corollary 3.17 with Corollary 3.20, we obtain the following corollary:

**Corollary 3.21.** *Let  $\nu$  be a finite Borel measure on  $(X, \nu_X)$  with  $m := \nu(X)$ . Put*

$$r_\kappa := \max \left\{ \text{Sep}(\nu; \kappa, C^{-2-\log_2 11}), \text{Sep} \left( \nu; \frac{m-\kappa}{3}, \frac{m-\kappa}{3} \right), \text{Sep} \left( \nu; \frac{m-\kappa}{3}, \kappa \right) \right\}$$

*for  $\kappa > 0$ . Then, there exists a point  $x_\kappa \in X$  such that  $\nu(B_X(x_\kappa, 3r_\kappa)) \geq 1 - \kappa$ . In particular, we have  $\text{diam}(\nu, m - \kappa) \leq 6r_\kappa$ .*

By using Corollary 3.21, we obtain the following proposition:

**Proposition 3.22.** *Assume that a compact metric group  $G$  continuously acts on  $(X, \nu_X)$ . Put*

$$\begin{aligned} r_{x,\kappa} := \max \left\{ \omega_x(+ \text{Sep}(G; \kappa, C^{-2-\log_2 11})), \omega_x \left( + \text{Sep} \left( G; \frac{1-\kappa}{3}, \frac{1-\kappa}{3} \right) \right), \right. \\ \left. \omega_x \left( + \text{Sep} \left( G; \frac{1-\kappa}{3}, \kappa \right) \right) \right\} \end{aligned}$$

*for  $x \in X$  and  $\kappa > 0$ . Then, for any  $\kappa \in (0, 1/2)$ , there exists a point  $z_{x,\kappa} \in X$  such that*

$$d_X(z_{x,\kappa}, gz_{x,\kappa}) \leq 3r_{x,\kappa} + \rho(3r_{x,\kappa})$$

for any  $g \in G$ . There also exists a point  $z'_{x,\kappa} \in Gx$  such that

$$d_X(z'_{x,\kappa}, gz'_{x,\kappa}) \leq \min\{6r_{x,\kappa} + \rho(6r_{x,\kappa}), 6r_{x,\kappa} + 2\rho(3r_{x,\kappa})\}$$

for any  $g \in G$ .

**Corollary 3.23.** *Assume that a compact metric group  $G$  continuously acts on an  $n$ -dimensional complete Riemannian manifold  $M$  with nonnegative Ricci curvature. Put*

$$r_\kappa := \max \left\{ \omega_x(+\text{Sep}(G; \kappa, 2^{-(2+\log_2 11)^n})), \omega_x\left(+\text{Sep}\left(G; \frac{1-\kappa}{3}, \frac{1-\kappa}{3}\right)\right), \right. \\ \left. \omega_x\left(+\text{Sep}\left(G; \frac{1-\kappa}{3}, \kappa\right)\right) \right\}$$

for  $x \in M$  and  $\kappa > 0$ . Then, for any  $x \in M$  and  $\kappa \in (0, 1/2)$ , there exists a point  $z_{x,\kappa} \in M$  such that

$$d_M(z_{x,\kappa}, gz_{x,\kappa}) \leq 3r_{x,\kappa} + \rho(3r_{x,\kappa})$$

for any  $g \in G$ . There also exists a point  $z'_{x,\kappa} \in Gx$  such that

$$d_M(z'_{x,\kappa}, gz'_{x,\kappa}) \leq \min\{6r_{x,\kappa} + \rho(6r_{x,\kappa}), 6r_{x,\kappa} + 2\rho(3r_{x,\kappa})\}$$

for any  $g \in G$ .

**3.5. Case of metric graphs.** In this subsection, we treat the case where  $X$  is a metric graph. Let  $\Gamma = (V, E)$  be a (possibly infinite) undirected connected combinatorial graph, that is,  $\Gamma$  is a 1-dimensional cell complex with the set  $V$  of vertices and the set  $E$  of edges. We allow the graph  $\Gamma$  to have multiple edges and loops. For vertices  $v, w \in V$  which are endpoints of an edge, we assign a positive number  $a_{v,w}$  such that  $a_\Gamma := \inf_{v' \neq w'} a_{v'w'} > 0$ . Every edge is identified with a bounded closed interval or a circle in  $\mathbb{R}^2$  with length  $a_{vw}$ , where  $v$  and  $w$  are endpoints of the edge. We then define the distance between two points in  $\Gamma$  to be the infimum of the length of paths joining them. The graph  $\Gamma$  together with such a distance function is called a *metric graph*.

**Lemma 3.24.** *Let  $(C, d_C)$  be a circle in  $\mathbb{R}^2$  with the Riemannian distance function  $d_C$  and  $\nu$  a finite Borel measure on  $C$  with  $m := \nu(C)$ . Then, for any  $\kappa > 0$ , we have*

$$\text{diam}(\nu, m - \kappa) \leq \frac{\pi}{\sqrt{2}} \text{Sep}\left(\nu; \frac{\kappa}{4}, \frac{\kappa}{4}\right)$$

*Proof.* Note that

$$d_{\mathbb{R}^2}(x, y) \leq d_C(x, y) \leq \frac{\pi}{2} d_{\mathbb{R}^2}(x, y)$$

for any  $x, y \in C$ . Denoting by  $\text{pr}_i : \mathbb{R}^2 \ni (x_1, x_2) \mapsto x_i \in \mathbb{R}$  the projection, by using Lemma 2.11, we therefore obtain

$$\begin{aligned}
\text{diam}(\nu, m - \kappa) &= \text{diam}(\nu|_{(C, d_C)}, m - \kappa) \\
&\leq \frac{\pi}{2} \text{diam}(\nu|_{(C, d_{\mathbb{R}^2})}, m - \kappa) \\
&\leq \frac{\pi}{\sqrt{2}} \max_{i=1,2} \text{diam} \left( (\text{pr}_i)_*(\nu|_{(C, d_{\mathbb{R}^2})}), m - \frac{\kappa}{2} \right) \\
&\leq \frac{\pi}{\sqrt{2}} \max_{i=1,2} \text{Sep} \left( (\text{pr}_i)_*(\nu|_{(C, d_{\mathbb{R}^2})}); \frac{\kappa}{4}, \frac{\kappa}{4} \right) \\
&\leq \frac{\pi}{\sqrt{2}} \text{Sep} \left( \nu|_{(C, d_{\mathbb{R}^2})}; \frac{\kappa}{4}, \frac{\kappa}{4} \right) \\
&\leq \frac{\pi}{\sqrt{2}} \text{Sep} \left( \nu; \frac{\kappa}{4}, \frac{\kappa}{4} \right).
\end{aligned}$$

This completes the proof.  $\square$

For every edge  $e \in E$  and  $r > 0$ , we put  $e_{-r} := \{x \in e \mid d_\Gamma(e, v) > r \text{ and } d_\Gamma(e, w) > r\}$ , where  $v$  and  $w$  are endpoints of the edge  $e$ .

**Theorem 3.25.** *Let  $\nu$  be a finite Borel measure on a metric graph  $\Gamma$  with  $m := \nu(\Gamma)$ . Assume that postive numbers  $a, \kappa, \kappa'$  satisfy that  $\kappa' < \kappa$ ,  $a < a_\Gamma$ , and*

$$\max \left\{ 2 \text{Sep} \left( \nu; \frac{\kappa}{3}, \frac{\kappa}{3} \right), 4 \text{Sep} \left( \nu; \frac{m - \kappa}{3}, \kappa' \right) \right\} < a$$

*Then, we have*

$$(3.3) \quad \text{diam}(\nu, m - \kappa) \leq \max \left\{ \frac{a}{2} + 2 \text{Sep} \left( \nu; \frac{\kappa}{3}, \kappa \right), \frac{\pi}{\sqrt{2}} \text{Sep} \left( \nu; \frac{\kappa - \kappa'}{4}, \frac{\kappa - \kappa'}{4} \right) \right\}.$$

*Proof.* We first consider the case of  $\nu(\bigcup_{v \in V} B_X(v, a/4)) \geq \kappa$ . Since  $\text{Sep}(\nu; \kappa/3, \kappa/3) < a/2$ , as in the proof of Claim 3.14, there exists a vertex  $v \in V$  such that  $\nu(B_X(v, a/4)) \geq \kappa/3$ . We thus obtain  $\nu(B_X(v, a/4 + \text{Sep}(\nu; \kappa/3, \kappa/3))) \geq m - \kappa$ , which implies (3.3).

We consider the other case that  $\nu(X \setminus \bigcup_{v \in V} B_X(v, a/4)) > m - \kappa$ . By the same method of Claim 3.14, either the following (1) or (2) holds:

- (1) There exists an edge  $e \in E$  such that  $e$  is not a loop and  $\nu(e_{-a/4}) \geq (m - \kappa)/3$ .
- (2) There exists a loop  $\ell \in E$  with  $\nu(\ell_{-a/4}) \geq (m - \kappa)/3$ .

If (1) holds, combining the same proof of Claim 3.13 with  $\text{Sep}(\nu; \kappa/3, \kappa') < a/4$ , we then have  $\nu(e) \geq m - \kappa'$ . We therefore obtain

$$\begin{aligned}
\text{diam}(\nu, m - \kappa) &\leq \text{diam}(\nu|_e, m - \kappa) \\
&= \text{diam}(\nu|_e, \nu(e) - (\nu(e) - m + \kappa)) \\
&\leq \text{Sep} \left( \nu|_e; \frac{\nu(e) - m + \kappa}{2}, \frac{\nu(e) - m + \kappa}{2} \right) \\
&\leq \text{Sep} \left( \nu; \frac{\kappa - \kappa'}{2}, \frac{\kappa - \kappa'}{2} \right).
\end{aligned}$$

If (2) holds, by Claim 3.13 and  $\text{Sep}(\nu; \kappa/3, \kappa') < a/4$ , we then get  $\nu(\ell) \geq m - \kappa'$ . Applying Lemma 3.24, we therefore obtain

$$\begin{aligned}
\text{diam}(\nu, m - \kappa) &\leq \text{diam}(\nu|_\ell, m - \kappa) \\
&= \text{diam}(\nu|_\ell, \nu(\ell) - (\nu(\ell) - m + \kappa)) \\
&\leq \frac{\pi}{\sqrt{2}} \text{Sep}\left(\nu|_\ell; \frac{\nu(\ell) - m + \kappa}{4}, \frac{\nu(\ell) - m + \kappa}{4}\right) \\
&\leq \frac{\pi}{\sqrt{2}} \text{Sep}\left(\nu|_\ell; \frac{\kappa - \kappa'}{4}, \frac{\kappa - \kappa'}{4}\right) \\
&\leq \frac{\pi}{\sqrt{2}} \text{Sep}\left(\nu; \frac{\kappa - \kappa'}{4}, \frac{\kappa - \kappa'}{4}\right).
\end{aligned}$$

This completes the proof of the theorem.  $\square$

**Corollary 3.26.** *Let  $\{X_n\}_{n=1}^\infty$  be a Lévy family and  $\Gamma$  a metric graph. Then, for any  $\kappa > 0$ , we have*

$$\lim_{n \rightarrow \infty} \text{ObsDiam}_\Gamma(X_n; -\kappa) = 0.$$

By virtue of Theorem 3.25, we obtain the following:

**Proposition 3.27.** *Assume that a compact metric group  $G$  continuously acts on a metric graph  $\Gamma$ . We also assume that positive numbers  $a, \kappa, \kappa'$  and a point  $x \in X$  satisfy that  $\kappa' < \kappa$ ,  $a < a_\Gamma$ , and*

$$\max\left\{2\omega_x\left(+\text{Sep}\left(G; \frac{\kappa}{3}, \frac{\kappa}{3}\right)\right), 4\omega_x\left(+\text{Sep}\left(G; \frac{1-\kappa}{3}, \kappa'\right)\right)\right\} < a.$$

Put

$$s_{x, \kappa, \kappa'} := \max\left\{\frac{a}{2} + 2\omega_x\left(+\text{Sep}\left(G; \frac{\kappa}{3}, \kappa\right)\right), \frac{\pi}{\sqrt{2}}\omega_x\left(+\text{Sep}\left(G; \frac{\kappa - \kappa'}{4}, \frac{\kappa - \kappa'}{4}\right)\right)\right\}.$$

Then, there exists a point  $z_{x, \kappa, \kappa'} \in Gx$  such that

$$d_X(z_{x, \kappa, \kappa'}, gz_{x, \kappa, \kappa'}) \leq s_{x, \kappa, \kappa'} + \rho(s_{x, \kappa, \kappa'})$$

for any  $g \in G$ .

**3.6. Case of Hadamard manifolds.** In this subsection, we consider the case where  $X$  is a Hadamard manifold  $N$ , i.e., a complete simply connected Riemannian manifold with nonpositive sectional curvature. The following theorem was obtained in [3, Theorem 1.3].

**Theorem 3.28.** *Let  $\{X_n\}_{n=1}^\infty$  be a Lévy family and  $N$  a Hadamard manifold. Then, for any  $\kappa > 0$ , we have*

$$\lim_{n \rightarrow \infty} \text{ObsDiam}_N(X_n; -\kappa) = 0.$$

3.6.1. *Central radius.* Let  $N$  be a Hadamard manifold. For a finite Borel measure on  $N$  with compact support, we indicate the center of mass of the measure  $\nu$  by  $c(\nu)$ . Given any  $\kappa > 0$ , putting  $m := \nu(N)$ , we define the *central radius*  $\text{CRad}(\nu, m - \kappa)$  of  $\nu$  as the infimum of  $\rho > 0$  such that  $\nu(B_N(c(\nu), \rho)) \geq m - \kappa$ .

**Proposition 3.29** (cf. [23, Proposition 5.4]). *For a finite Borel measure  $\nu$  on  $\mathbb{R}^k$  with the compact support, we have*

$$c(\nu) = \frac{1}{\nu(\mathbb{R}^k)} \int_{\mathbb{R}^k} x d\nu(x).$$

**Proposition 3.30** (cf. [23, Proposition 5.10]). *Let  $N$  be a Hadamard manifold and  $\nu$  a finite Borel measure on  $N$  with the compact support. Then,  $x = c(\nu)$  if and only if*

$$\int_N \exp_x^{-1}(y) d\nu(y) = 0.$$

*In particular, identifying the tangent space of  $N$  at the point  $c(\nu)$  with the Euclidean space of the same dimension of  $N$ , we have  $c((\exp_{c(\nu)}^{-1})_*(\nu)) = 0$ .*

Proposition 2.13 directly implies the following corollary:

**Corollary 3.31.** *Assume that a compact metric group acts on a Hadamard manifold  $N$  and put  $r_x := \lim_{\kappa \uparrow 1/2} \text{CRad}(\nu_{G,x}, 1 - \kappa)$  for  $x \in X$ . Then, we have*

$$d_X(c(\nu_{G,x}), gc(\nu_{G,x})) \leq r_x + \rho(+r_x)$$

*for any  $g \in G$ . Moreover, there exists a point  $z_x \in Gx$  such that*

$$d_X(z_x, gz_x) \leq \min\{2r_x + \rho(+2r_x), 2r_x + 2\rho(+r_x)\}$$

*for any  $g \in G$ .*

3.6.2. *Hölder actions.* In this subsection, we consider a Hölder action of a compact Lie group to a Hadamard manifold.

Let a compact Lie group  $G$  acts on a Hadamard manifold  $N$ . We shall consider the case where  $\omega_x(\eta) \leq C_1 \eta^\alpha$  holds for some  $x \in N$  and  $C_1, \alpha > 0$ .

Combining Gromov's observation in [8, Section 13] with one in [10, Section 3 $\frac{1}{2}$ .41], we obtain the following theorem:

**Theorem 3.32.** *Let  $M$  be a compact Riemannian manifold and  $N$  be a Hadamard manifold. Assume that a continuous map  $f : M \rightarrow N$  satisfies that*

$$d_N(f(x), f(y)) \leq C_1 d_M(x, y)^\alpha$$

*for some  $C_1 > 0$ ,  $\alpha > 1$ , and all  $x, y \in M$ . Then, the map  $f : M \rightarrow N$  is a constant map.*

*Proof.* Put  $\mathbb{E}(f) := c(f_*(\mu_M))$ . We shall prove that  $\text{Supp } f_*(\mu_X) = \{\mathbb{E}(f)\}$ , which implies the theorem. Suppose that  $\text{Supp } f_*(\mu_X) \neq \{\mathbb{E}(f)\}$ . We identify the tangent space of  $N$  at  $\mathbb{E}(f)$  with the Euclidean space  $\mathbb{R}^k$ , where  $k$  is the dimension of  $N$ . According to the hinge theorem (see [22, Chapter IV, Remark 2.6]), the map  $\exp_{\mathbb{E}(f)}^{-1} : N \rightarrow \mathbb{R}^k$  is 1-Lipschitz.



Since the map  $\exp_{\mathbb{E}(f)}^{-1}$  is isometric on rays issuing from  $\mathbb{E}(f)$  and  $\text{Supp } f_*(\mu_M) \neq \{\mathbb{E}(f)\}$ , we have

$$\int_M |(\exp_{\mathbb{E}(f)}^{-1} \circ f)(x)|^2 d\mu_M(x) = \int_M d_N(f(x), \mathbb{E}(f))^2 d\mu_M(x) > 0.$$

Denoting by  $((\exp_{\mathbb{E}(f)}^{-1} \circ f)(x))_i$  the  $i$ -th component of  $(\exp_{\mathbb{E}(f)}^{-1} \circ f)(x)$ , we hence see that there exists  $i_0$  such that

$$\int_M |((\exp_{\mathbb{E}(f)}^{-1} \circ f)(x))_{i_0}|^2 d\mu_M(x) > 0.$$

Putting  $\varphi := (\exp_{\mathbb{E}(f)}^{-1} \circ f)_{i_0}$ , we observe that

$$\|\text{grad}_x \varphi\| = \limsup_{y \rightarrow x} \frac{|\varphi(y) - \varphi(x)|}{d_M(y, x)} \leq \limsup_{y \rightarrow x} \frac{C_1 d_M(y, x)^\alpha}{d_M(y, x)} = 0$$

and the function  $\varphi$  has mean zero by Proposition 3.30. We therefore obtain

$$0 < \lambda_1(M) = \inf \frac{\int_M \|\text{grad}_x g\|^2 d\mu_M(x)}{\int_M g(x)^2 d\mu_M(x)} \leq \frac{\int_M \|\text{grad}_x \varphi\|^2 d\mu_M(x)}{\int_M \varphi(x)^2 d\mu_M(x)} = 0,$$

where the infimum is taken over all nontrivial Lipschitz maps  $g : M \rightarrow \mathbb{R}$  with mean zero. This is a contradiction. This completes the proof.  $\square$

**Corollary 3.33.** *Assume that a compact Lie group  $G$  continuously acts on a Hadamard manifold  $N$ . We also assume that there exists a point  $x \in X$  such that the condition  $\omega_x(\eta) \leq C_1 \eta^\alpha$  holds for some  $\alpha > 1$ . Then, the point  $x$  is a fixed point.*

Assume that a compact metric group  $G$  continuously acts on a Hadamard manifold  $N$ . In view of Corollary 3.33, we shall consider the case of  $0 < \alpha \leq 1$ .

We assume that a compact metric group  $G$  satisfies that

$$(3.4) \quad \alpha_G(r) \leq C_2 e^{-C_3 r^\beta} \text{ for some } C_2, C_3, \beta > 0.$$

See Examples 2.2 and 2.3 for examples.

Let a compact metric group continuously acts on a metric space  $X$ . For any  $r > 0$  and  $x \in X$ , we define  $\omega_x^{-1}(r)$  as the infimum of  $d_G(g, g')$ , where  $g$  and  $g'$  run over all elements in  $G$  such that  $d_X(gx, g'x) \geq r$ .

**Lemma 3.34.** *Assume that a compact metric group continuously acts on a metric space  $X$ . Then, for any  $x \in X$ , we have*

$$\alpha_{(X, \nu_{G,x})}(r) \leq \alpha_G(\omega_x^{-1}(r)).$$

*Proof.* Let  $A \subseteq X$  be any Borel subset such that  $\nu_{G,x}(A) \geq 1/2$ . From the definition of  $\omega_x^{-1}(r)$ , we get

$$\{g \in G \mid gx \in A\}_{+\omega_x^{-1}(r)} \subseteq \{g \in G \mid gx \in A_{+r}\}.$$

Since  $\mu_G(\{g \in G \mid gx \in A\}) \geq 1/2$ , we hence obtain

$$\nu_{G,x}(X \setminus A_{+r}) \leq \mu_G(G \setminus \{g \in G \mid gx \in A\}_{+\omega_x^{-1}(r)}) \leq \alpha_G(\omega_x^{-1}(r)).$$

This completes the proof.  $\square$

**Lemma 3.35.** *Let a compact metric group  $G$  continuously acts on a metric space  $X$ . Assume that a point  $x \in X$  satisfies the following Hölder condition:*

$$(3.5) \quad \omega_x(\eta) \leq C_1 \eta^\alpha \text{ holds for some } C_1 > 0 \text{ and } 0 < \alpha \leq 1.$$

We also assume that the group  $G$  satisfies the condition (3.4). Then, we have

$$\alpha_{(N, \nu_{G,x})}(r) \leq C_2 e^{-C_1^{-\beta/\alpha} C_3 r^{\beta/\alpha}}.$$

*Proof.* By the assumption (3.5),  $d_X(gx, g'x) > C_1 s^\alpha$  implies that  $d_G(g, g') > s$ , that is,  $d_X(gx, g'x) \geq r$  yields that  $d_G(g, g') \geq (r/C_1)^{1/\alpha}$ . We hence get  $\omega_x^{-1}(r) \geq (r/C_1)^{1/\alpha}$ . By using this and Lemma 3.34, we obtain

$$\alpha_{(X, \nu_{G,x})}(r) \leq \alpha_G(\omega_x^{-1}(r)) \leq \alpha_G((r/C_1)^{1/\alpha}) \leq C_2 e^{-C_1^{-\beta/\alpha} C_3 r^{\beta/\alpha}}.$$

This completes the proof.  $\square$

We denote by  $\gamma_k$  the standard Gaussian measure on  $\mathbb{R}^k$  with density  $(2\pi)^{-k/2} e^{-|x|^2/2}$ . For any  $p \geq 0$ , we put

$$M_p := \int_{\mathbb{R}} |s|^p d\gamma_1(s) = 2^{p/2} \pi^{-1/2} \Gamma\left(\frac{p+1}{2}\right).$$

The same proof of [12, Theorem 1] implies the following theorem:

**Theorem 3.36** (cf. [12, Theorem 1]). *Assume that an mm-space  $X$  satisfies that  $\alpha_X(r) \leq C_1 e^{-C_2 r^p}$  for some  $C_1, C_2 > 0$  and some  $p \geq 1$ . Then, for any 1-Lipschitz function  $f : X \rightarrow \mathbb{R}^k$  with mean zero, we have*

$$\int_X |f(x)|^p d\mu_X(x) \leq \frac{C}{C_2 M_p} \int_{\mathbb{R}^k} |y|^p d\gamma_k(y) = \frac{C}{C_2 M_p} \cdot \frac{2^{p/2} \Gamma(\frac{p+k}{2})}{\Gamma(\frac{k}{2})} \approx \frac{C k^{p/2}}{C_2},$$

where  $C$  is a constant depending only on  $p$  and  $C_1$ .

**Theorem 3.37.** *Let a compact metric group  $G$  continuously acts on a  $k$ -dimensional Hadamard manifold  $N$ . Assume that a point  $x \in N$  satisfies the Hölder condition (3.5). We also assume that the group  $G$  satisfies (3.4) and  $\alpha \leq \beta$ . Then, there exists a point  $z_x \in Gx$  such that*

$$(3.6) \quad \text{diam}(Gz_x) \leq \frac{CC_1 k^{1/2}}{(C_3)^{\alpha/\beta}} + \rho\left(\frac{CC_1 k^{1/2}}{(C_3)^{\alpha/\beta}}\right),$$

where  $C$  is a constant depending only on  $\alpha/\beta$  and  $C_1$ .

*Proof.* To apply Corollary 3.31, we shall estimate  $\text{CRad}(\nu_{G,x}, 1 - \kappa)$  for  $0 < \kappa < 1/2$  from the above. Putting  $z := c(\nu_{G,x})$ , as in the proof of Theorem 3.32, we identify the

tangent space of  $N$  at  $z$  with the Euclidean space  $\mathbb{R}^k$ . Since the map  $\exp_z^{-1} : N \rightarrow \mathbb{R}^k$  is a 1-Lipschitz map, by virtue of Lemma 3.35 and Theorem 3.36, we have

$$\int_N d_N(y, z)^{\beta/\alpha} d\nu_{G,x}(y) = \int_N |(\exp_z^{-1})(y)|^{\beta/\alpha} d\nu_{G,x}(y) \leq \frac{CC_1^{\beta/\alpha} k^{\beta/(2\alpha)}}{C_3},$$

where  $C$  is a constant depending only on  $C_2$  and  $\beta/\alpha$ . Combining this inequality with the Chebyshev inequality, we hence get

$$\text{CRad}(\nu_{G,x}, 1 - \kappa) \leq \frac{CC_1 k^{1/2}}{(C_3 \kappa)^{\alpha/\beta}}$$

for any  $0 < \kappa$ . Applying Corollary 3.31, we therefore obtain (3.6). This completes the proof.  $\square$

**3.6.3. Cases of finite groups.** In this subsubsection, we shall consider the case where  $G$  is a finite group. Let  $G$  be a finite group and  $S \subseteq G \setminus \{e_G\}$  be a symmetric set of generators of  $G$ . We denote by  $\Gamma(G, S)$  the *Cayley graph* of  $G$  with respect to  $S$ . For such  $S$ , we shall consider the group  $G$  as a metric group with respect to the Cayley graph distance function.

Let  $\Gamma = (V, E)$  be a simple finite graph, where *simple* means that there is at most one edge joining two vertices and no loops from a vertex to itself. The discrete Laplacian  $\Delta_\Gamma$  act on functions  $f$  on  $V$  as follows

$$\Delta_\Gamma f(x) := \sum_{y \sim x} (f(x) - f(y)),$$

where  $x \sim y$  means that  $x$  and  $y$  are connected by an edge. We denote by  $\lambda_1(\Gamma)$  the non-zero first eigenvalue of the Laplacian  $\Delta_\Gamma$ .

As Theorem 3.32, Gromov's observation in [8, Section 13] together with one in [10, Section 3 $\frac{1}{2}$ .41] imply the following lemma:

**Lemma 3.38.** *Let  $S \subseteq G \setminus \{e_G\}$  be a symmetric set of generators of a finite group  $G$  and assume that the group  $G$  continuously acts on a  $k$ -dimensional Hadamard manifold  $N$ . Then, for any  $x \in N$  and  $\kappa > 0$ , we have*

$$\text{CRad}(\nu_{G,x}, 1 - \kappa) \leq \omega_x(1) \left( \frac{k \# S}{2\kappa \lambda_1(\Gamma(G, S))} \right)^{1/2}.$$

*Proof.* Suppose that

$$(3.7) \quad r := \text{CRad}(\nu_{G,x}, 1 - \kappa) > \omega_x(1) \left( \frac{k \# S}{2\kappa \lambda_1(\Gamma(G, S))} \right)^{1/2}.$$

As in the proof of Theorem 3.32, we identify the tangent space of  $N$  at  $z := c(\nu_{G,x})$  with the Euclidean space  $\mathbb{R}^k$ . By the Chebyshev inequality, we get

$$\int_G |(\exp_z^{-1} \circ f^x)(g)|^2 d\mu_G(g) = \int_G d_N(f^x(g), z)^2 d\mu_G(g) \geq \kappa r^2.$$

Hence, there exists  $i_0$  such that

$$(3.8) \quad \int_G ((\exp_z^{-1} \circ f^x)(g))_{i_0}^2 d\mu_G(g) \geq \frac{\kappa r^2}{k}.$$

Putting  $\varphi := (\exp_z^{-1} \circ f^x)_{i_0}$ , by (3.7) and (3.8), we obtain

$$\begin{aligned} \lambda_1(\Gamma(G, S)) &= \inf \frac{\sum_{g, g' \in G; g \sim g'} (f(g) - f(g'))^2}{2 \sum_{g \in G} f(g)^2} \\ &\leq \frac{\sum_{g, g' \in G; g \sim g'} (\varphi(g) - \varphi(g'))^2}{2 \sum_{g \in G} \varphi(g)^2} \\ &\leq \frac{\sum_{g, g' \in G; g \sim g'} d_N(f^x(g), f^x(g'))^2}{2 \sum_{g \in G} \varphi(g)^2} \\ &\leq \frac{\#G \#S \cdot \omega_x(1)^2}{\#G \int_G \varphi(g)^2 d\mu_G(g)} \\ &= \frac{\omega_x(1)^2 \#S}{\int_G \varphi(g)^2 d\mu_G(g)} \\ &\leq \frac{\omega_x(1)^2 k \#S}{\kappa r^2} \\ &< \lambda_1(\Gamma(G, S)), \end{aligned}$$

where the infimum is taken over all nontrivial functions  $f : G \rightarrow \mathbb{R}$  such that  $\sum_{g \in G} f(g) = 0$ . This is a contradiction. This completes the proof.  $\square$

Applying Lemma 3.38 to Corollary 3.31, we obtain the following theorem:

**Theorem 3.39.** *Let  $S \subseteq G \setminus \{e_G\}$  be a symmetric set of generators of a finite group  $G$  and assume that the group  $G$  continuously acts on a  $k$ -dimensional Hadamard manifold  $N$ . Then, for any  $x \in N$ , we have*

$$d_N(c(\nu_{G,x}), gc(\nu_{G,x})) \leq \omega_x(1) \left( \frac{k \#S}{\lambda_1(\Gamma(G, S))} \right)^{1/2} + \rho \left( + \omega_x(1) \left( \frac{k \#S}{\lambda_1(\Gamma(G, S))} \right)^{1/2} \right)$$

for any  $g \in G$ . There also exists a point  $z_x \in Gx$  such that

$$\begin{aligned} d_N(z_x, gz_x) &\leq \min \left\{ \omega_x(1) \left( \frac{k \#S}{\lambda_1(\Gamma(G, S))} \right)^{1/2} + \rho \left( + 2\omega_x(1) \left( \frac{k \#S}{\lambda_1(\Gamma(G, S))} \right)^{1/2} \right), \right. \\ &\quad \left. \omega_x(1) \left( \frac{k \#S}{\lambda_1(\Gamma(G, S))} \right)^{1/2} + 2\rho \left( + \omega_x(1) \left( \frac{k \#S}{\lambda_1(\Gamma(G, S))} \right)^{1/2} \right) \right\} \end{aligned}$$

for any  $g \in G$ .

#### 4. LÉVY GROUP ACTION

In this section, we discuss about a Lévy group action to concrete metric spaces appeared in Section 3.

A metrizable group  $G$  is called a *Lévy group* if it contains an increasing chain of compact subgroups  $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots$  having an everywhere dense union in  $G$  and such that for some right-invariant compatible distance function  $d_G$  on  $G$  the groups  $G_n$ ,  $n \in \mathbb{N}$ , equipped with the Haar measures  $\mu_{G_n}$  normalized as  $\mu_{G_n}(G_n) = 1$  and the restrictions of the distance function  $d_G$ , form a Lévy family. See [7], [17], [18], [19] and references therein for informations about a Lévy group.

Let a topological group  $G$  acts on a metric space  $X$ . The action is called *bounded* if for any  $\varepsilon > 0$  there exists a neighbourhood  $U$  of the identity element  $e_G \in G$  such that  $d_X(x, gx) < \varepsilon$  for any  $g \in U$  and  $x \in X$ . Note that every bounded action is continuous.

**Lemma 4.1** (cf. [19, Theorem 1]). *Assume that a metric group  $G$  with a right invariant distance function  $d_G$  boundedly acts on a metric space  $X$ . Then, orbit maps  $f_x : G \rightarrow X$  for all  $x \in X$  are uniformly equicontinuous.*

We shall consider an action of a Lévy group to a metric space  $X$  satisfying the following condition:

( $\diamond$ ): We have  $\lim_{n \rightarrow \infty} \text{ObsDiam}_X(X_n; -\kappa) = 0$  for any  $\kappa > 0$  and any Lévy family  $\{X_n\}_{n=1}^\infty$ .

Note that  $\mathbb{R}$ -trees, doubling spaces, metric graphs, and Hadamard manifolds satisfy the condition ( $\diamond$ ) (see Section 3).

**Conjecture 4.2.** *Any complete Riemannian manifolds satisfy the condition( $\diamond$ ).*

Let a topological group  $G$  acts on a metric space  $X$ . We say that the topological group  $G$  acts on  $X$  *by uniform isomorphisms* if for each  $g \in G$ , the map  $X \ni x \mapsto gx \in X$  is uniform continuous. The action is said to be *uniformly equicontinuous* if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $d_X(gx, gy) < \varepsilon$  for every  $g \in G$  and  $x, y \in X$  with  $d_X(x, y) < \delta$ . Given a subset  $S \subseteq G$  and  $x \in X$ , we put  $Sx := \{gx \mid g \in S\}$ .

**Proposition 4.3.** *Assume that a Lévy group  $G$  boundedly acts on a metric space  $X$  having the property ( $\diamond$ ) by uniform isomorphisms. Then for any compact subset  $K \subseteq G$  and any  $\varepsilon > 0$ , there exists a point  $x_{\varepsilon, K} \in X$  such that  $\text{diam}(Kx_{\varepsilon, K}) \leq \varepsilon$ .*

**Proposition 4.4.** *There are no non-trivial bounded uniformly equicontinuous actions of a Lévy group on a metric space having the property ( $\diamond$ ).*

*Proof of Propositions 4.3 and 4.4.* From the definition of  $G$ , the group  $G$  contains an increasing chain of compact subgroups  $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots$  having an everywhere dense union in  $G$  such that for some right-invariant compatible distance function  $d_G$  on  $G$ , the sequence  $\{(X, d_X, \mu_{G_n})\}_{n=1}^\infty$  forms a Lévy family. Let  $x \in X$  be an arbitrary point.

We first prove Proposition 4.3. Since  $G$  boundedly acts on  $X$  and  $d_G$  is right-invariant, by virtue of Lemma 4.1, for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $d_X(gy, g'y) < \varepsilon/2$  for any  $y \in X$  and  $g, g' \in G$  with  $d_G(g, g') \leq \delta$ . Take a subset  $\{g_1, g_2, \dots, g_N\} \subseteq G$  such that each  $g \in K$  is within distance  $\delta$  of the set  $\{g_1, g_2, \dots, g_N\}$  and all  $g_i$  are contained in  $G_\ell$  for some large  $\ell \in \mathbb{N}$ . Since the orbit map  $f_x : G \rightarrow X$  is uniformly continuous, by using Corollary 2.17, the sequence  $\{(X, d_X, \nu_{G_n, x})\}_{n=1}^\infty$  is a Lévy family. From the

property  $(\diamond)$  of the space  $X$  the identity maps  $\text{id}_n : (X, d_X, \nu_{G_n, x}) \rightarrow X$  concentrate, that is,  $\lim_{n \rightarrow \infty} \text{diam}(\nu_{G_n, x}, 1 - \kappa) = 0$  for any  $\kappa > 0$ . Hence there exist  $\varepsilon_n > 0$  and  $x_n \in X_n$  such that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  and  $\lim_{n \rightarrow \infty} \nu_{G_n, x}(B_X(x_n, \varepsilon_n)) = 1$ . Take  $n_0 \in \mathbb{N}$  such that  $n_0 \in \mathbb{N}$ ,  $\nu_{G_{n_0}, x}(B_X(x_{n_0}, \varepsilon_{n_0})) > 1/2$  and  $\varepsilon_{n_0} \leq \rho^{\{g_1, g_2, \dots, g_N\}, X}(\varepsilon_{n_0}) < \varepsilon/4$ . The same method of the proof of (2.1), we obtain

$$d_X(x_{n_0}, g_i x_{n_0}) \leq \varepsilon_{n_0} + \rho^{\{g_1, g_2, \dots, g_N\}, X}(\varepsilon_{n_0}) < \varepsilon/2$$

for any  $g_i$ . For any  $g \in K$ , choosing  $g_i$  with  $d_G(g_i, g) < \delta$ , we obtain

$$d_X(x_{n_0}, g x_{n_0}) \leq d_X(x_{n_0}, g_i x_{n_0}) + d_X(g_i x_{n_0}, g x_{n_0}) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

by the definition of  $\delta > 0$ . This completes the proof of Proposition 4.3.

We next prove Proposition 4.4. Since  $\lim_{\eta \rightarrow 0} \rho^{(G, X)}(\eta) = 0$ , by using Corollary 2.15, we get

$$\text{diam}(G_n x) \leq 2 \lim_{\kappa \uparrow 1/2} \text{diam}(\nu_{G_n, x}, 1 - \kappa) + 2\rho^{(G, X)}\left(+ \lim_{\kappa \uparrow 1/2} \text{diam}(\nu_{G_n, x}, 1 - \kappa)\right) \rightarrow 0$$

as  $n \rightarrow \infty$ . Since  $G_1 x \subseteq G_2 x \subseteq \dots \subseteq G_n x \subseteq G_{n+1} x \subseteq \dots$ , we therefore obtain  $G_n x = \{x\}$  for any  $n \in \mathbb{N}$ . This completes the proof of Proposition 4.4.  $\square$

Note that every continuous action of a topological group on a compact metric space is bounded. Since a compact metric space has the property  $(\diamond)$  and a Lévy group  $G$  contains an increasing chain of compact subgroups  $G_n$  having an everywhere dense union, Proposition 4.3 includes the fixed point theorem ([7, Theorem 7.1]) by Gromov and Milman.

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